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SMALL OSCILLATIONS OF AN ORTHOTROPIC CYLINDRICAL SHELL CONTAINING A LIQUID COVERED BY RIGID ENDPLATES*

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The oscillation problem for a Timoshenko-type orthotropic cylindrical shell containing a liquid covered by two movable, rigid endplates is considered. The problem is solved by decomposing the required function in a modified Fourier series. Numerous papers /1 – 6/ have been devoted to the study of oscillations of a finite cylindrical shell containing a liquid bounded by a single endplate. The case of a short shell with two endplates has also been investigated in /7 – 11/. Here the motion of the shell is described by classical equations, and the shell material itself is assumed to be isotropic.

1. An orthotropic cylindrical shell of length $2l_0$ and radius R is covered at its edges by absolutely rigid plane endplates and entirely filled with an acoustic liquid. Within the shell pressure oscillates according to the law $q_0 e^{i\omega_0 t}$. In a dimensionless cylindrical coordinate system r, θ, x, whose x-axis coincides with the shell axis and whose origin is placed at the center of the shell, axisymmetric motion of the shell may be described by two differential equations relative to the displacement components u, w /12 – 14/:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \tau^2}\right) u + \nu_{21} \frac{\partial w}{\partial x} = 0 \tag{1.1}$$

$$\left[\frac{\partial^4}{\partial x^4} - (1+\varepsilon_1)\frac{\partial^4}{\partial x^2 \partial \tau^2} - m\varepsilon_1 \frac{\partial^2}{\partial x^2} + \varepsilon_1 \frac{\partial^4}{\partial \tau^4} + (m\varepsilon_1 + k^{-2})\frac{\partial^2}{\partial \tau^2} + mk^{-2}\right] w + m\nu_{12}\left(\varepsilon_1 \frac{\partial^3}{\partial x \partial \tau^2} - \varepsilon_1 \frac{\partial^3}{\partial x^3} + k^{-2}\frac{\partial}{\partial x}\right) u + \frac{\rho}{k_0}\left(\varepsilon_1 \frac{\partial^3}{\partial \tau^3} - \varepsilon_1 \frac{\partial^3}{\partial x^2 \partial \tau} + k^{-2}\frac{\partial}{\partial \tau}\right)[\varphi]_{r=1} = 2q(k^{-2} - \varepsilon_1 \omega^2) e^{i\omega\tau} \tag{1.2}$$

$$q = \frac{1 - \nu_{12}\nu_{21}}{2k_0 E_1} q_0, \quad \omega = \frac{R}{a_1}\omega_1, \quad a_1 = \frac{E_1}{G_{12}(1 - \nu_{12}\nu_{21})}$$

$$k_0 = \frac{2h}{R}, \quad m = \frac{E_2}{E_1}, \quad \rho = \frac{\rho_0}{\rho_1}, \quad k^2 = \frac{1}{12}k_0^2, \quad \tau = \frac{c_1}{R_1} t$$

$$\varepsilon_1 = \frac{E_1}{k' G_{13}(1 - \nu_{12}\nu_{21})}, \quad c_1^2 = \frac{E_1}{\rho_1 (1 - \nu_{12}\nu_{21})}, \quad \varphi = \frac{\varphi_1}{R_1 c_1}$$

Here φ_1 is the velocity potential describing the motion of the liquid; t time; ρ_0 and ρ_1 density of liquid and material of the shell, respectively; ν_{12} and ν_{21} Poisson coefficients of the orthotropic material of the shell; $2h$ thickness of shell; k' shear coefficient; and E_1, E_2 G_{12} tensile and shear moduli of elasticity. The modulus G_{13} serves for taking into account the anisotropy of the elastic properties across the thickness of the shell.

The dimensionless velocity potential φ of an acoustic liquid satisfies the wave equation (c_0 is the speed of sound in a liquid)

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} = \frac{1}{\mu^2}\frac{\partial^2 \varphi}{\partial \tau^2}, \quad \mu = \frac{c_0}{c_1} \tag{1.3}$$

A continuous motion condition must hold on the wetted surface of the shell, and hinge support conditions at the edges of the shell:

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=1} = \frac{\partial w}{\partial \tau}; \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \left(x = \pm l, \quad l = \frac{l_0}{R}\right) \tag{1.4}$$

In addition, at the edges of the shell it is necessary to take into account the transfer of pressure to the end faces from the liquid, the inducing pressure, and the force of inertia of the endplates. This condition has the form (h_* and ρ_* are the thickness and density of the material of the endplates)

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$$\frac{\partial u}{\partial x}+\nu_{21}w+Q\frac{\partial^2 u}{\partial\tau^2}-qe^{i\omega\tau}+\frac{\rho}{k_0}\int_0^1\frac{\partial\varphi}{\partial\tau}r\,dr=0\quad(x=\pm l)\tag{1.5}$$

$$Q=\frac{h_*\rho_*}{4h\rho_1}$$

Besides these conditions, there is also a condition specifying continuous motion of the liquid on the endplates, which in the present case may be written as an integral

$$2\int_0^1\frac{\partial\varphi}{\partial x}r\,dr=\frac{\partial u}{\partial\tau}\quad(x=\pm l)\tag{1.6}$$

The last equality indicates that the instantaneous flow rate of the liquid across each extreme section of the shell is equal to the volume freed by the corresponding endplate in this time. Such a replacement of the exact condition by an integral condition will not lead to major errors /15/.

2. To solve the problem, we represent the displacement components of the points of the shell in the form (A_{jn} are unknown coefficients)

$$u=U(x)e^{i\omega\tau}\tag{2.1}$$

$$w=\sum_{j=1}^{2}\sum_{n=1}^{\infty}A_{jn}\xi_j(\alpha_{jn}x)e^{i\omega\tau}\tag{2.2}$$

$$\xi_1(x)=\cos x,\quad \xi_2(x)=\sin x,\quad \alpha_{1n}=\frac{\pi}{l}(n-0.5),\quad \alpha_{2n}=\frac{\pi}{l}n$$

In Sect.3 we will show that the decomposition (2.2) is valid.

In such a formulation, the last two boundary conditions in (1.4) are satisfied indentically. Substituting expressions (2.1) and (2.2) in (1.1), we obtain a second-order ordinary differential equation that may be used to determine the function $U(x)$. Its solution may be written in the form

$$U(x)=D_1\sin(\omega x)+D_2\cos(\omega x)+\nu_{21}\sum_{j=1}^{2}\sum_{n=1}^{\infty}\frac{\alpha_{jn}}{\varkappa_{jn}}A_{jn}\xi_j'(\alpha_{jn}x)\quad(\omega\neq\alpha_{jn})\tag{2.3}$$

$$U(x)=D_1\sin(\omega x)+D_2\cos(\omega x)-\frac{\nu_{21}}{2}A_{im}x\xi_i(\omega x)+\nu_{21}\sum_{j=1}^{2}\sum_{n=1}^{\infty}{}_0\frac{\alpha_{jn}}{\varkappa_{jn}}A_{jn}\xi_j'(\alpha_{jn}x)\quad(\omega=\alpha_{im})$$

Here $\varkappa_{jn}=d_{jn}^2-\omega^2$; D_1 and D_2 unkown constants, with the "prime" denoting differentiation with respect to the independent variable, while the zero subscript in the final sum denotes that $j\neq i$ and $n\neq m$ simultaneously.

The wave equation (1.3), first boundary condition (1.4) and boundedness condition on the axis will be satisfied if the velocity potential of the liquid is used in the form

$$\varphi=\sum_{j=1}^{2}\sum_{n=1}^{\infty}A_{jn}\psi(\beta_{jn}r)\xi_j(\alpha_{jn}x)i\omega e^{i\omega\tau},\quad \beta_{jn}^2=\alpha_{jn}^2-\frac{\omega^2}{\mu^2}\tag{2.4}$$

$$\psi(\beta_{jn}r)=\frac{I_0(\beta_{jn}r)}{\beta_{jn}I_1(\beta_{jn})}\quad(\omega<\alpha_{jn}\mu)$$

$$\psi(\beta_{jn}r)=-\frac{J_0(i\beta_{jn}r)}{i\beta_{jn}J_1(i\beta_{jn})}\quad(\omega>\alpha_{jn}\mu)$$

$$\psi(\beta_{jn}r)=C=\mathrm{const}\quad(\omega=\alpha_{jn}\mu)$$

(J_n and I_n are n-th order Bessel functions of a real and imaginary variable). In the special case when $\beta_{jn}^2=0$, the corresponding decomposition coefficient in (2.2) $A_{jn}=0$.

If the boundary conditions (1.5) and (1.6) are satisfied, we can then find a relation between the coefficients D_1, D_2, A_{1n}, and A_{2n}. Three such cases are possible:

1°. $\omega\neq\alpha_{jn}$ $(j=1,\ 2)$

$$D_j=\frac{(-1)^j}{\xi_j'(\omega l)}\sum_{n=1}^{\infty}(-1)^n\alpha_{jn}d_{jn}A_{jn}\tag{2.5}$$

$$A_{2n}=b_nA_{1n}-\eta_nq_n;\quad q_1=q_2=q,\quad q_k=0\quad(k=3,\ 4,\ \dots)$$

$$d_{jn}=\frac{2}{\beta_{jn}}-\frac{\nu_{21}}{\varkappa_{jn}}, \quad b_{2m-1}=(-1)^m\frac{\alpha_{1m}\beta_{2,2m-1}^2}{\alpha_{2,2m-1}\beta_{1m}^2 B_m}\times$$
$$(4Q\omega^2+d_{1m}d_{2,2m}\beta_{1m}^2\beta_{2,2m}^2)\varkappa_{2,2m}\varkappa_{2,2m-1}\operatorname{ctg}(\omega l)$$
$$b_{2m}=(-1)^m\frac{\alpha_{1m}\beta_{2,2m}^2}{\alpha_{2,2m}\beta_{1m}^2 B_m}(4Q^2\omega^2+d_{1m}d_{2,2m-1}\beta_{1m}^2\beta_{2,2m-1}^2)\times\varkappa_{2,2m}\varkappa_{2,2m-1}\operatorname{ctg}(\omega l)$$
$$\eta_{2m}=\gamma_m\frac{d_{2,2m-1}}{\alpha_{2,2m}}, \quad \eta_{2m-1}=\gamma_m\frac{d_{2,2m}}{\alpha_{2,2m-1}}$$
$$B_m=2Q\omega\nu_{21}(\beta_{2,2m}^2\varkappa_{2,2m-1}-\beta_{2,2m-1}^2\varkappa_{2m})$$
$$\gamma_m=\varkappa_{2,2m}\varkappa_{2,2m-1}\beta_{2,2m}^2\beta_{2,2m-1}^2/(\omega B_m)$$

2°. $\omega=\alpha_{1p}$

$$D_1=\sum_{\substack{n=1\\(n\neq p)}}^{\infty}A_{1n}\alpha_{1n}d_{1n}(-1)^{p+n+1}, \quad D_2=\frac{\nu_{21}}{2}A_{1p}l+2Q\omega\sum_{\substack{n=1\\(n\neq p)}}^{\infty}A_{1n}\frac{\alpha_{1n}}{\beta_{1n}^2}, \quad A_{2n}=a_n q_n$$

$$a_{2m-1}=\frac{d_{2,2m}\varkappa_{2,2m}\varkappa_{2,2m-1}}{Q\omega^2\nu_{21}\alpha_{2,2m-1}(d_{2,2m-1}\varkappa_{2,2m-1}-d_{2,2m}\varkappa_{2,2m})}, \quad a_{2m}=\frac{d_{2,2m-1}\varkappa_{2,2m-1}\varkappa_{2,2m}}{Q\omega^2\nu_{21}\alpha_{2,2m}(d_{2,2m}\varkappa_{2,2m}-d_{2,2m-1}\varkappa_{2,2m-1})}$$

3°. $\omega=\alpha_{2p}$

$$D_1=(-1)^p\frac{q}{\omega}+2Q\omega(-1)^p\sum_{\substack{n=1\\(n\neq p)}}^{\infty}(-1)^n\frac{\alpha_{2n}}{\beta_{2n}^2}A_{2n}$$

$$D_2=(-1)^p\sum_{\substack{n=1\\(n\neq p)}}^{\infty}(-1)^n\alpha_{2n}d_{2n}A_{2n}, \quad A_{1n}=0$$

Substituting the resulting expressions for the displacement components of the points in the shell and the velocity potential of the liquid in light of (2.5) in the motion equation of the shell (1.2) and applying the Bubnov–Galerkin method to this equation, we obtain an infinite system of linear algebraic equations for the decomposition coefficients A_{1n} or A_{2n}:

$$[f(\alpha_{1s})+b_s f(\alpha_{2s})]A_{1s}+2\frac{m\nu_{12}\omega}{k^2}\sum_{n=1}^{\infty}A_{1n}\left[\frac{2Q\omega\alpha_{1n}\alpha_{2s}}{\beta_{1n}^2\varkappa_{2s}}-\right. \tag{2.6}$$
$$\left.\frac{\alpha_{1s}^2}{\varkappa_{1s}}d_{1n}\operatorname{ctg}(\omega l)\right](-1)^{s+n}=4(-1)^s\frac{\varepsilon_1\omega^2-k^{-2}}{\alpha_{1s}}q+\eta_s q_s f(\alpha_{2s})$$
$$(\omega\neq\alpha_{jp})$$
$$f(\alpha_{1s})A_{1s}-2\frac{m\nu_{12}\omega}{k^2}\sum_{\substack{n=1\\(n\neq p)}}^{\infty}(-1)^{n+s}A_{1n}\frac{\alpha_{1n}\alpha_{2s}}{\varkappa_{2s}}d_{1n}=$$
$$4\frac{(-1)^s}{\alpha_{1s}}(\varepsilon_1\omega^2-k^{-2})q-a_s q_s\ (s\neq p,\omega=\alpha_{1p})$$
$$f(\alpha_{2s})A_{2s}+2\frac{m\nu_{12}\omega}{k^2}\sum_{\substack{n=1\\(n\neq p)}}^{\infty}(-1)^{n+s}A_{2n}\frac{\alpha_{1s}\alpha_{2n}}{\varkappa_{1s}}d_{2n}=$$
$$4(-1)^s\frac{q}{\alpha_{1s}}(\varepsilon_1\omega^2-k^{-2})\quad(s\neq p,\omega=\alpha_{2p})$$
$$f(\alpha_{js})=\left\{\alpha_{js}^4-(1+\varepsilon_1)\omega^2\alpha_{js}^2+m\varepsilon_1\alpha_{js}^2+\varepsilon_1\omega^4+mk^{-2}-\right.$$
$$(m\varepsilon_1+k^{-2})\omega^2-(k^{-2}+\varepsilon_1\alpha_{js}^2-\varepsilon_1\omega^2)\left[\frac{\alpha_{js}^2}{\varkappa_{js}}m\nu_{12}\nu_{21}+\right.$$
$$\left.\left.\frac{\rho}{k_0}\psi(\beta_{js})\right]\right\}l;\quad \psi(\beta_{js})=\psi(\beta_{js}r)|_{r=1};\quad j=1,2;s=1,2,3,\ldots$$

If we set $q = 0$, a solution of the frequency and modeshape problem of the natural oscillations of a cylindrical shell covered by rigid endplates and completely filled with a liquid may be obtained from the above solution. In this case, the equation for determining the eigenfrequencies ω may be represented in the form of an infinite determinant (δ_{ns} is the Kronecker symbol)

$$\det (M_{ns}) = 0 \quad (n,\ s = 1,\ 2,\ \ldots) \tag{2.7}$$

$$M_{ns} = [f(\alpha_{1n}) + b_n f(\alpha_{2n})]\,\delta_{ns} + 2\,\frac{m\nu_{12}\omega}{k^2}\left[2Q\omega\,\frac{\alpha_{1n}\alpha_{2s}}{\beta_{1n}^2\varkappa_{2s}} - \frac{\alpha_{1s}^2}{\varkappa_{1s}}\,d_{1n}\,\mathrm{ctg}\,(\omega l)\right](-1)^{s+n}, \quad (\omega \neq \alpha_{jp})$$

When $\omega = \alpha_{jp}$, the coefficients M_{sn} may be written down without any difficulty. The solution was studied by means of a numerical method.

The computations were performed for the case of an isotropic ($\nu_{12} = \nu_{21} = \nu = 0.3$; $E_1 = 2G_{13}(1+\nu)$; $m = 1$) and orthotropic ($\nu_{12} = 0.3$; $\nu_{21} = 0.15$; $E_1 = 40G_{13}$; $m = 0.5$) shell for the following general data: $\mu = 0.5$; $k' = 5/6$; $\rho = \rho_0/\rho_1 = 1/7.8$; $2l = 2l_0/R = 4$; $Q = h_* \rho_* / (4h\rho) = 2$. In calculating the natural oscillations, the coefficients of the decomposition were normalized in such a way that $A_{11} = 1$. A system of ten equations was solved numerically. The frequencies of the natural oscillations were found to correct within 10^{-3}.

Table 1

Shell	Q	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
Orthotropic	2	0.392	0.733	0.792	0.899	0.950	1.086
Isotropic	2	0.393	0.787	1.011	1.065	1.178	1.216
Isotropic	4	0.393	0.745	1.011	1.058	1.178	1.188

The first six eigenfrequencies are presented in the accompanying Table 1 for the case of isotropic and orthotropic shells filled with a compressible liquid. In the last row may be found the dimensionless specific mass of the endplate $Q = 4$. From an analysis of the numerical results, it follows that the orthotropic material of the shell has little effect on the first eigenfrequency, but substantially changes subsequent frequencies. The mass of the endplates noticeable changes the second eigenfrequency. Clearly, at this frequency longitudinal oscillations of the shell predominate.

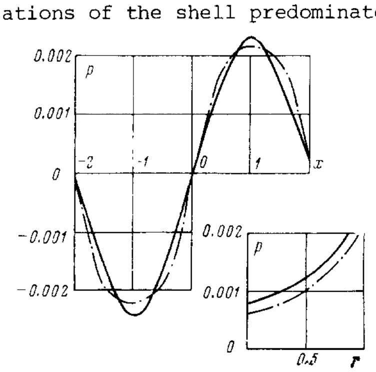

Fig.1

In the accompanying Fig.1 may be found an example showing how the amplitude of the excess liquid pressure $p = -\rho \times \partial\varphi / \partial\tau$ varies in the case of a shell with length $2l = 4$ at $x = 1$ along the radius r, and at $r = 1$ along the x-axis. The studies were conducted at a frequency of the induced oscillations $\omega = 0.1$. The solid curves correspond to an orthotropic shell, and the dot-and-dash curves, to an isotropic shell.

The above analysis demonstrates that the pressure near the walls of the shell is always greater than the pressure along the axis, increasing whenever the frequency of the induced oscillations approaches the eigenfrequency of the system. The decomposition coefficients A_{jn} $(j = 1, 2)$ in the case of both the natural and induced oscillations rapidly decrease with increasing ordinal number n, a result which indicates that the solution has a high degree of convergence.

3. Theorem. If the function $f(x)$ with period 2π is piecewise-differentiable in the closed interval $[-\pi, \pi]$, and vanishes at the edges of the interval, it may be expanded in a series of the form

$$f(x) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(n - \frac{1}{2}\right)x + b_n \sin nx \right\} \tag{3.1}$$

which converges at every point $x_0 \in [-\pi, \pi]$ and has the sum

$$S_0(x_0) = \frac{1}{2}\,[f(x_0 + 0) + f(x_0 - 0)] \tag{3.2}$$

As the function is continuous at the point x_0, the sum of the series (3.1) is equal to the value of the function at this point.

Proof. It may be verifed that the system of functions $\cos(n-1/2)x$, $\sin nx$ $(n=1,2,\ldots)$ is orthogonal on the closed interval $[-\pi,\pi]$, and that the coefficients of the expansion a_n and b_n are related to $f(x)$ as

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos\left(n-\frac{1}{2}\right)u\,du,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du \tag{3.3}$$

It is self-evident that the theorem holds at the endpoints of the interval $(x_0=\pm\pi)$. Let us prove it for an arbitrary point $x_0\in\,]-\pi,\pi[$. The partial sum of the series (3.1) at the point x_0 is equal

$$S_k(x_0)=\sum_{n=1}^{k}[a_n\cos(n-0.5)x_0+b_n\sin nx_0] \tag{3.4}$$

Substituting the values of the coefficients a_n and b_n from (3.3) in (3.4) and taking into account the identities

$$\sum_{n=1}^{k}\cos\left(n-\frac{1}{2}\right)t-\frac{2}{2\sin t/2}\sum_{n=1}^{k}[\sin nt-\sin(n-1)t]=\frac{\sin kt}{2\sin t/2}, \tag{3.5}$$

$$\frac{1}{2}+\sum_{n=1}^{k}\cos nt=\frac{1}{2\sin t/2}\left\{\sin t/2+\sum_{n=1}^{k}\left[\sin\left(n+\frac{1}{2}\right)t-\sin\left(n-\frac{1}{2}\right)t\right]\right\}=\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin t/2}$$

we obtain

$$S_k(x_0)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\left[\frac{\sin k(u+x_0)}{2\sin[(u+x_0)/2]}+\frac{\sin k(u-x_0)}{2\sin[(u-x_0)/2]}+\right. \tag{3.6}$$
$$\left.\frac{\sin\left(k+\frac{1}{2}\right)(u-x_0)}{2\sin[(u-x_0)/2]}-\frac{\sin\left(k+\frac{1}{2}\right)(u+x_0)}{2\sin[(u+x_0)/2]}\right]du$$

Performing the substitution $t=u-x_0$ and $t=u+x_0$ in the integrand (3.6) and bearing in mind the fact that the function $f(x)$ is periodic, we obtain

$$S_k(x_0)=\frac{1}{2\pi}\int_{0}^{\pi}[f(-x_0+t)+f(-x_0-t)]\left[\frac{\sin\left(k+\frac{1}{2}\right)t\left(\cos\frac{t}{2}-1\right)}{2\sin t/2}-\right. \tag{3.7}$$
$$\left.\frac{1}{2}\cos\left(n+\frac{1}{2}\right)t\right]dt+\frac{1}{\pi}\int_{\pi}^{\pi-x_0}[f(-t-x_0)+f(t+x_0)]\frac{\sin kt}{2\sin t/2}\,dt$$

Below we will require the values of certain integrals that may be obtained from the identities in (3.5):

$$\int_{-\pi}^{\pi}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin t/2}\,dt=\pi \tag{3.8}$$

$$\int_{-\pi}^{\pi}\frac{\sin kt}{2\sin t/2}\,dt=\sum_{n=1}^{k}\frac{4(-1)^{n+1}}{2n-1}=\pi+\varepsilon\quad\left(\lim_{k\to\infty}\varepsilon=0\right) \tag{3.9}$$

We may verify that (3.9) is valid if we set $x=1$ in the MacLaurin series for the function $\operatorname{arc\,tg} x$.

From (3.8) and (3.9), and in light of the fact that the integrand functions are even, we find that

$$1=\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin kt+\sin\left(k+\frac{1}{2}\right)t}{2\sin t/2}\,dt-\frac{\varepsilon}{2\pi} \tag{3.10}$$

Multiplying equalities (3.10) and (3.2) and subtracting the result obtained from (3.7), we find

$$S_k(x_0)-S_0(x_0)=\frac{1}{2\pi}\int_{0}^{\pi}q_1(t)\sin kt\,dt=\frac{1}{2\pi}\int_{0}^{\pi}g_1(t)\sin\left(k+\frac{1}{2}\right)t\,dt+ \tag{3.11}$$

$$\frac{1}{2\pi}\int_0^{\pi} g_2(t)\sin\left(k+\frac{1}{2}\right)t\,dt-\frac{1}{4\pi}\int_0^{\pi} g_3(t)\cos\left(k+\frac{1}{2}\right)t\,dt+$$
$$\frac{1}{2\pi}\int_{\pi}^{\pi-x_0} g_4(t)\sin kt\,dt+\varepsilon\,\frac{f(x_0+0)+f(x_0-0)}{4\pi}$$
$$g_1(t)=\left[\frac{f(x_0+t)-f(x_0+0)}{t}-\frac{f(x_0-t)-f(x_0-0)}{-t}\right]\frac{t/2}{\sin t/2}$$
$$g_2(t)=[f(t-x_0)+f(-t-x_0)]\,\frac{\cos(t/2)-1}{2\sin t/2}$$

$$g_3(t)=f(t-x_0)+f(-t-x_0),\ g_4(t)=[f(-t-x_0)+f(t+x_0)]\,\frac{1}{\sin t/2}$$

The functions g_1, g_2 and g_3 are piecewise-continuous on the interval $[0,\pi]$, since the function $f(t)$ is piecewise-continuous. The behavior of these functions as $t\to+0$ remains an open question. It may verified that

$$\lim g_1(t)=f'(x_0+0)-f'(x_0-0),\ \ \lim g_2(t)=0,\ \ \lim g_3(t)=f(-x_0+0)+f(-x_0-0),\ t\to+0$$

The function $g_4(t)$ will be piecewise-continuous on the interval $[\pi,\pi-x_0]$ for arbitrary $x_0\in\,]-\pi,\ \pi[$.

If we use the Riemann theorem /16/, we find from (3.11) that as $k\to\infty$,

$$\lim_{k\to\infty} S_k(x_0)-S_0(x_0)=0$$

The theorem is proved.

In the case of an arbitrary interval $[-l,\ l]$, we have

$$f(x)=\sum_{n=1}^{\infty}\left[a_n\cos\frac{\pi\left(n-\frac{1}{2}\right)x}{l}+b_n\sin\frac{\pi nx}{l}\right]$$
$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi\left(n-\frac{1}{2}\right)x}{l}\,dx,\quad b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi nx}{l}\,dx$$

The even function may be expanded in a series only in cosines, while the odd function, in sines.

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